

# Noncommutative Instantons on the 4–Sphere from Quantum Groups

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## Abstract

We describe an approach to the noncommutative instantons on the 4–sphere based on quantum group theory. We quantize the Hopf bundle  $\mathbb{S}^7 \rightarrow \mathbb{S}^4$  making use of the concept of quantum coisotropic subgroups. The analysis of the semiclassical Poisson–Lie structure of  $U(4)$  shows that the diagonal  $SU(2)$  must be conjugated to be properly quantized. The quantum coisotropic subgroup we obtain is the standard  $SU_q(2)$ ; it determines a new deformation of the 4–sphere  $\Sigma_q^4$  as the algebra of coinvariants in  $\mathbb{S}_q^7$ . We show that the quantum vector bundle associated to the fundamental corepresentation of  $SU_q(2)$  is finitely generated and projective and we compute the explicit projector. We give the unitary representations of  $\Sigma_q^4$ , we define two 0–summable Fredholm modules and we compute the Chern–Connes pairing between the projector and their characters. It comes out that even the zero class in cyclic homology is non trivial.

## 1 Introduction

Since the work [25] on instantons on noncommutative  $\mathbb{R}^4$  a lot of attention has been devoted to the problem of gauge theories on noncommutative four manifolds. In ordinary differential geometry, the topological properties of instantons in  $\mathbb{R}^4$  are better understood by studying fibre bundles on the sphere  $\mathbb{S}^4$ . In noncommutative geometry this is not an easy task: it is more natural to define the problem directly on the noncommutative sphere.

Very recently, in [10] and [12] two different deformations of  $\mathbb{S}^4$  were proposed. The one in [10] preserves the property of having zero the first Chern class which

is not trivial in [12]. In this second case the deformation is a suspension of the quantum 3–sphere  $SU_q(2)$  obtained by adding a central generator.

In this paper we propose an alternative approach, based more directly on quantum groups and on Hopf algebraic techniques.

In noncommutative geometry finitely generated projective modules, *i.e.* the quantum vector bundles, are the central object to develop gauge theories. From this point of view there is no obvious notion of structure group. Quantum groups provide a construction of quantum vector bundles which is closer to ordinary differential geometry. The first attempts go back to [14], [26] and [6], where the gauge theory is developed starting from the notion of Hopf–Galois extension, which is the analogue of principal bundles in the Hopf algebra setting, see [28]. The associated quantum vector bundles have a Hopf algebra on the fiber and, if they admit a connection, are finitely generated and projective modules [11].

Although this definition works in principle, it is not enough to explain all known interesting examples. This problem is better understood if we concentrate on the specific class of principal bundles given by homogeneous spaces. A quantum homogeneous space is an example of Hopf–Galois extension only if it is obtained as quotient by a quantum subgroup (*i.e.* a Hopf algebra quotient). But quantum subgroups are very rare. For instance between the quantum 2–spheres introduced by Podleś in [27] only one, the standard one, is such an example. It is necessary to generalize the notion of subgroup, allowing a more general quotient procedure. This is possible by using *quantum coisotropic subgroups*: they are quotient by a coideal, right (or left) ideal, so that they inherit only the coalgebra, while the algebra structure is weakened to a right (or left) module. Their semiclassical interpretation is illuminating: in a Poisson–Lie group every Poisson (resp. coisotropic) subgroup can be quantized to a quantum (resp. coisotropic) subgroup (see [8]). Nevertheless conjugation, which does not change topology, can break Poisson properties: for instance a subgroup conjugated to a Poisson subgroup can be only coisotropic or can have no Poisson properties at all (see for instance  $SL(2, \mathbb{R})$  in [3]). Coisotropic subgroups can be quantized and give rise to inequivalent quantum homogeneous spaces: for instance all the Podleś quantum spheres are obtained as quotient of coisotropic  $U(1)$ . The general scheme to describe such examples could be the so called  $C$ –Galois extensions, see for instance [5, 4].

The principal bundle on  $\mathbb{S}^4$  corresponding to  $SU(2)$  instantons with charge  $-1$  has  $\mathbb{S}^7 = U(3) \backslash U(4)$  as total space and the action on the fibre is obtained by considering  $SU(2)$  as diagonal subgroup of  $U(4)$ . In this description  $\mathbb{S}^4$  is the double coset  $U(3) \backslash U(4) / SU(2)$ . In the quantum setting, odd spheres were obtained in [30] as homogeneous spaces of  $U_q(N)$  with respect to the quantum subgroup  $U_q(N-1)$  so that the left quotient is easily quantized. The right quotient is more problematic because the diagonal  $SU(2)$  doesn't survive in the quantization of  $U(4)$ ; indeed the analysis of the limit Poisson structure on  $U(4)$  shows that it is not coisotropic. We then have to look for coisotropic subgroups in the conjugacy class of the diagonal one. It comes out that there is at least one which defines what we call the Poisson Hopf

bundle in  $\mathbb{S}^4$  (Proposition 3). In this bundle, which is topologically equivalent to the usual Hopf bundle, both the total and the base spaces are Poisson manifolds and the projection is a Poisson map. Its quantization is straightforward: the quantum coisotropic subgroup turns out to be equivalent as coalgebra to  $SU_q(2)$  (Proposition 4) and the algebra of functions over the quantum 4–sphere  $\Sigma_q^4$  is then obtained as the subalgebra of coinvariants in  $\mathbb{S}_q^7$  with respect to this  $SU_q(2)$  (Proposition 6). This deformation of the algebra of functions on  $\mathbb{S}^4$  is different from those introduced in [10] and [12]. We then study the quantum vector bundle associated to the fundamental corepresentation of  $SU_q(2)$  and give the explicit projector (Proposition 7). We describe the unitary irreducible representations of  $\Sigma_q^4$  (Equation 7 and 8); there is a 1–dimensional representation and an infinite dimensional one realized by trace class operators (Proposition 8). Finally we study the Chern class in cyclic homology of the projector and compute the Chern–Connes pairing with a trace induced by the trace class representation (Proposition 10). It comes out that, on the contrary with [10] and [12], they are all non trivial. This result is the analogue of what was obtained in [22, 18] for the standard Podleś 2–sphere.

## 2 Quantization of Coisotropic Subgroups

A Poisson–Lie group  $(\mathcal{G}, \{, \})$  is a Lie group  $\mathcal{G}$  with a Poisson bracket  $\{, \}$  such that the multiplication map  $m : \mathcal{G} \times \mathcal{G} \rightarrow \mathcal{G}$  is a Poisson map with respect to the product Poisson structure in  $\mathcal{G} \times \mathcal{G}$ . The Poisson bracket  $\{, \}$  is identified by a bivector  $\omega$  (*i.e.* a section of  $\bigwedge^2 T\mathcal{G}$ ) such that  $\{\phi_1, \phi_2\}(x) = \omega(x)(d_x\phi_1, d_x\phi_2)$ . (For more details see [7] and [29])

Every Poisson–Lie group induces a natural bialgebra structure on  $\mathfrak{g} = \text{Lie}(\mathcal{G})$  which will be called the tangent bialgebra of  $\mathcal{G}$ . Indeed,  $\delta : \mathfrak{g} \rightarrow \mathfrak{g} \wedge \mathfrak{g}$  is defined by  $\langle X, d_e\{f, g\} \rangle = \langle \delta(X), f \otimes g \rangle$ , where  $X \in \mathfrak{g}$  and  $f, g \in C^\infty(\mathcal{G})$ .

The point we want to discuss is the behaviour of subgroups and corresponding homogeneous spaces with respect to the Poisson structure. A Lie subgroup  $\mathcal{H}$  of  $\mathcal{G}$  is called a Poisson–Lie subgroup if it is also a Poisson submanifold of  $\mathcal{G}$ , *i.e.* if the immersion map  $\iota : \mathcal{H} \rightarrow \mathcal{G}$  is a Poisson map. There are various characterizations for such subgroups: as invariant subspaces for the dressing action or as union of symplectic leaves [21].

The property of being a Poisson–Lie subgroup is, evidently, a very strong one. We need then to characterize a family of subgroups satisfying weaker hypothesis with respect to the Poisson structure.

In Poisson geometry a submanifold  $N$  of a Poisson manifold  $(M, \omega)$  is said to be coisotropic if  $\omega|_{\text{Ann}(T_x N)} = 0$ , where  $\text{Ann}(T_x N) = \{\alpha \in T_x^*(M) \mid \alpha(v) = 0 \quad \forall v \in T_x N\}$ . Coisotropy can be formulated very neatly as an algebraic property at the function algebra level (see [29]). Indeed a locally closed submanifold  $N$  of the Poisson manifold  $(M, \omega)$  is coisotropic if and only if for every  $f, g \in C^\infty(M)$

$$f|_N = 0, g|_N = 0 \Rightarrow \{f, g\}|_N = 0 \quad .$$

Thus locally closed coisotropic submanifolds correspond to manifolds whose defining ideal is not a Poisson ideal but only a Poisson subalgebra.

A Lie subgroup  $\mathcal{H}$  of  $(\mathcal{G}, \omega)$  is said a *coisotropic subgroup* if it is coisotropic as Poisson submanifold. In the connected case there are nice characterizations, as shown for example in [20]; we will need the following one:

**Proposition 1** *A connected subgroup  $\mathcal{H}$  of  $(\mathcal{G}, \omega)$  with  $\mathfrak{h} = \text{Lie}\mathcal{H}$  is coisotropic iff  $\delta(\mathfrak{h}) \subset \mathfrak{g} \wedge \mathfrak{h}$  and it is Poisson–Lie iff  $\delta(\mathfrak{h}) \subset \mathfrak{h} \wedge \mathfrak{h}$ .*

Given a Poisson–Lie group  $\mathcal{G}$  and a coisotropic subgroup  $\mathcal{H}$  the natural projection map  $\mathcal{G} \rightarrow \mathcal{H}\backslash\mathcal{G}$  coinduces a Poisson structure on the quotient. If  $\mathcal{K}$  is a second subgroup of  $\mathcal{G}$  a condition which guarantees that even the projection on the double coset is Poisson is given by the following:

**Proposition 2 ([20])** *Let  $(M, \omega_o)$  be a Poisson manifold with a Poisson action of a Poisson–Lie group  $(\mathcal{G}, \omega)$ . Let  $\mathcal{K}$  be a coisotropic connected subgroup of  $\mathcal{G}$ . If the orbit space  $M/\mathcal{K}$  is a manifold there exists a unique coinduced Poisson bracket such that the natural projection  $M \rightarrow M/\mathcal{K}$  is Poisson.*

We now recall how these concepts can be translated in a Hopf algebra setting, (see [2, 8] for more details). Given a real quantum group  $(\mathcal{A}, *, \Delta, S, \epsilon)$  we will call *real coisotropic quantum right (left) subgroup*  $(\mathcal{K}, \tau_{\mathcal{K}})$  a coalgebra, right (left)  $\mathcal{A}$ –module  $\mathcal{K}$  such that:

- i) there exists a surjective linear map  $\pi : \mathcal{A} \rightarrow \mathcal{K}$ , which is a morphism of coalgebras and of  $\mathcal{A}$ –modules (where  $\mathcal{A}$  is considered as a module on itself via multiplication);
- ii) there exists an antilinear map  $\tau_{\mathcal{K}} : \mathcal{K} \rightarrow \mathcal{K}$  such that  $\tau_{\mathcal{K}} \circ \pi = \pi \circ \tau$ , where  $\tau = * \circ S$ .

A  $*$ –Hopf algebra  $\mathcal{S}$  is said to be a real quantum subgroup of  $\mathcal{A}$  if there exists a  $*$ –Hopf algebra epimorphism  $\pi : \mathcal{A} \rightarrow \mathcal{S}$ ; evidently this is a particular coisotropic subgroup. We remark that a coisotropic quantum subgroup is not in general a  $*$ –coalgebra but it has only an involution  $\tau_{\mathcal{K}}$  defined on it.

Right (left) coisotropic quantum subgroups are obviously characterized by the kernel of the projection, which is a  $\tau$ –invariant two–sided coideal, right (left) ideal in  $\mathcal{A}$ . It is easy to verify that if the kernel is also  $*$ –invariant then it is an ideal and the quotient is a real quantum subgroup.

A  $*$ –algebra  $B$  is said to be an embeddable quantum left (right)  $\mathcal{A}$ –homogeneous space if there exists a coaction  $\mu : B \rightarrow B \otimes \mathcal{A}$ , ( $\mu : B \rightarrow \mathcal{A} \otimes B$ ) and an injective morphism of  $*$ –algebras  $\iota : B \rightarrow \mathcal{A}$  such that  $\Delta \circ \iota = (\iota \otimes \text{id}) \circ \mu$  ( $\Delta \circ \iota = (\text{id} \otimes \iota) \circ \mu$ ).

Embeddable quantum homogeneous spaces can be obtained as the space of coinvariants with respect to the coaction of coisotropic quantum subgroups. For instance if  $\mathcal{K}$  is a right (left) subgroup and  $\Delta_\pi = (\text{id} \otimes \pi)\Delta$  ( ${}_\pi\Delta = (\pi \otimes \text{id})\Delta$ ), then

$$B^\pi = \{a \in \mathcal{A} \mid \Delta_\pi a = a \otimes \pi(1)\} \quad ({}^\pi B = \{a \in \mathcal{A} \mid {}_\pi\Delta a = \pi(1) \otimes a\}),$$

is an homogeneous space with  $\mu = \Delta$ .

If  $\rho : V \rightarrow \mathcal{K} \otimes V$  is a corepresentation of  $\mathcal{K}$ , we define the cotensor product as

$$\mathcal{A} \square_\rho V = \{F \in \mathcal{A} \otimes V \mid (\Delta_\pi \otimes \text{id}) F = (\text{id} \otimes \rho) F\}.$$

We have that  $\mathcal{A} \square_\rho V$  is a left  $B^\pi$ -module. Let  $\rho$  be unitary and  $\{e_i\}$  be an orthonormal basis of  $V$ ; if  $F = \sum_i F_i \otimes e_i$ , let's define  $\langle F, G \rangle = \sum_i F_i G_i^*$ . It is shown in [2] that  $\langle \cdot, \cdot \rangle$  is a sesquilinear form on  $\mathcal{A} \square_\rho V$  with values in  $B^\pi$ .

The correspondence between coisotropic quantum subgroups and embeddable quantum homogeneous spaces is bijective only provided some faithful flatness conditions on the module and comodule structures are satisfied (see [23] for more details).

The role of coisotropic subgroups can also be appreciated in the context of formal and algebraic equivariant quantization. While it is known that not every Poisson homogeneous space admits such quantization, it holds true that every quotient of a Poisson–Lie group by a coisotropic subgroup can be equivariantly quantized. Although such quotients do not exhaust the class of quantizable Poisson spaces they provide a large subclass in it. Furthermore in functorial quantization they correspond to embeddable quantum homogeneous spaces. More on the subject can be found in [15].

### 3 The Classical Instanton with $k = -1$

In this section we review the construction of the principal bundle corresponding to instantons with topological charge  $k = -1$  (see [1]). We denote with  $\mathbb{H}$  the quaternions generated by  $i, j, k$  with the usual relations  $i^2 = j^2 = k^2 = -1$ , and  $ij = -ji = k, jk = -kj = i, ki = -ik = j$ . The total space of the bundle is defined as  $E = \{(q_1, q_2) \in \mathbb{H}^2 \mid |q_1|^2 + |q_2|^2 = 1\}$ , the base space is  $P_1(\mathbb{H}) = \{[(q_1, q_2)] \mid (q_1, q_2) \simeq (q_1\lambda, q_2\lambda), (q_1, q_2) \in \mathbb{H}^2, \lambda \in \mathbb{H}\}$  and the bundle projection is  $p(q_1, q_2) = [(q_1, q_2)]$ . The fibre is  $SU(2)$  which acts on  $\mathbb{H}^2$  by the diagonal right multiplication of quaternions of unit modulus. The quaternionic polynomial functions  $B = \text{Pol}(P_1(\mathbb{H}))$  on the base space are generated by  $R = q_1\bar{q}_1$  and  $Q = q_1\bar{q}_2$ , with the relation  $|Q|^2 = R(1 - R)$ .

The fundamental representation of  $SU(2)$  can be realized again by right multiplication of unit quaternions on  $\mathbb{H}$ . The space  $\mathcal{E}$  of sections of the associated vector bundle is the space of equivariant functions  $s : E \rightarrow \mathbb{H}$ , *i.e.* such that  $s(q_1, q_2)\lambda = s(q_1\lambda, q_2\lambda)$ , for  $|\lambda| = 1$ . It is generated as a left  $B$ -module by  $s_1(q_1, q_2) = q_1$  and  $s_2(q_1, q_2) = q_2$  and it has an hermitian structure  $\langle \cdot, \cdot \rangle : \mathcal{E} \times \mathcal{E} \rightarrow B$  given by  $\langle s_1, s_2 \rangle = s_1\bar{s}_2$ .

We can define  $G \in M_2(B)$  with  $G_{ij} = \langle s_i, s_j \rangle$ . By direct computation we obtain that

$$G = G^2 = \begin{pmatrix} R & Q \\ \bar{Q} & 1 - R \end{pmatrix}. \quad (1)$$

It is easy then to verify that  $\mathcal{E} \simeq B^2 G$ .

For our future purposes, we have to describe this bundle in a Hopf algebraic language. We first remark that  $E$  is isomorphic to  $\mathbb{S}^7 = U(3) \backslash U(4)$  and  $P_1(\mathbb{H})$  to  $\mathbb{S}^4 = U(3) \backslash U(4) / SU(2)$ .

Let  $t_f = \{t_{ij}\}_{ij=1}^4$  define the fundamental representation of  $U(4)$ . Then  $\Delta(t_{ij}) = \sum_k t_{ik} \otimes t_{kj}$  and let  $\ell : \text{Pol}(U(4)) \rightarrow \text{Pol}(U(3))$  be the Hopf algebra projection defined by  $\ell(t_{4j}) = \ell(t_{j4}) = 0$  for  $j = 1, 2, 3$ , and  $\ell(t_{44}) = 1$ . The algebra of polynomial functions on  $\mathbb{S}^7$  is given by the coinvariants  ${}^\ell\text{Pol}(U(4))$  and it is generated by  $z_i = t_{4i}$  with the relation  $\sum_i |z_i|^2 = 1$ . Let  $r : \text{Pol}(U(4)) \rightarrow \text{Pol}(SU(2))$  be the Hopf algebra projection defined by

$$r(t) = \begin{pmatrix} \alpha & \beta & 0 & 0 \\ -\beta^* & \alpha^* & 0 & 0 \\ 0 & 0 & \alpha & \beta \\ 0 & 0 & -\beta^* & \alpha^* \end{pmatrix}, \quad |\alpha|^2 + |\beta|^2 = 1.$$

As usual  $\text{Pol}(U(4)/SU(2))$  is obtained as the space of coinvariants  $\text{Pol}(U(4))^r$ . The algebra of polynomial functions on  $\mathbb{S}^4$  is  ${}^\ell\text{Pol}(U(4)) \cap \text{Pol}(U(4))^r$  and is generated by  $R = |z_1|^2 + |z_2|^2$ ,  $A = z_1 z_3^* + z_2 z_4^*$  and  $B = z_1 z_4 - z_2 z_3$ , with the relation  $|A|^2 + |B|^2 = R(1 - R)$ .

Let  $\tau_f : \mathbb{C}^2 \rightarrow \text{Pol}(SU(2)) \otimes \mathbb{C}^2$  be the fundamental corepresentation of  $\text{Pol}(SU(2))$

$$\tau_f \begin{pmatrix} e_1 \\ e_2 \end{pmatrix} = \begin{pmatrix} \alpha & \beta \\ -\beta^* & \alpha \end{pmatrix} \otimes \begin{pmatrix} e_1 \\ e_2 \end{pmatrix}.$$

The left  $\text{Pol}(\mathbb{S}^4)$ -module of sections of the associated vector bundle is obtained as  $\mathcal{E} = \text{Pol}(\mathbb{S}^7) \square_{\tau_f} \mathbb{C}^2$ . As a  $\text{Pol}(\mathbb{S}^4)$ -module,  $\mathcal{E}$  is generated by

$$f_1 = \begin{pmatrix} z_1 \\ z_2 \end{pmatrix}, \quad f_2 = \begin{pmatrix} z_2^* \\ -z_1^* \end{pmatrix}, \quad f_3 = \begin{pmatrix} z_3 \\ z_4 \end{pmatrix}, \quad f_4 = \begin{pmatrix} z_4^* \\ -z_3^* \end{pmatrix}.$$

With the usual hermitian structure we define  $G \in \text{Pol}(\mathbb{S}^4) \otimes M_4(\mathbb{C})$  with  $G_{ij} = \langle f_i, f_j \rangle$  and obtain that

$$G = G^2 = \begin{pmatrix} R & 0 & A & B \\ 0 & R & -B^* & A^* \\ A^* & -B & 1 - R & 0 \\ B^* & A & 0 & 1 - R \end{pmatrix}. \quad (2)$$

With the usual representation of  $\mathbb{H}$  as  $\mathbb{C}^2$ , where  $(z_1, z_2)$  is identified with  $z_1 + z_2 j$ , it is easy to verify that  $Q = A - B j$ ,  $f_1 = q_1$ ,  $f_2 = -j q_1$ ,  $f_3 = q_2$  and  $f_4 = -j q_2$ . Once we introduce the representation of the quaternions with Pauli matrices it is easy to verify that (1) and (2) define the same projector.

## 4 Poisson Hopf bundle on $\mathbb{S}^4$

Let us identify  $\mathfrak{g} = \mathbf{u}(4) = \text{Lie } U(4)$  with its defining representation by antihermitian  $4 \times 4$  matrices. The  $SU(2)$  generators of the Dynkin diagram are, for  $i = 1, 2, 3$

$$H_i = i(e_{ii} - e_{i+1,i+1}) , \quad E_i = \frac{1}{2i}(e_{i,i+1} + e_{i+1,i}) , \quad F_i = \frac{1}{2}(e_{i,i+1} - e_{i+1,i}) ,$$

where  $e_{ij}$  are the elementary matrices with entries  $(e_{ij})_{kl} = \delta_{ik}\delta_{jl}$  and the central generator is  $H = i\mathbb{I}$ . The Poisson–Lie structure of  $U(4)$  is defined by the canonical coboundary bialgebra given on these generators by

$$\delta_{\mathbb{R}}(H_i) = 0, \quad \delta_{\mathbb{R}}(H) = 0, \quad \delta_{\mathbb{R}}(E_i) = E_i \wedge H_i, \quad \delta_{\mathbb{R}}(F_i) = F_i \wedge H_i . \quad (3)$$

The generators  $h = \frac{1}{4}H_1 + \frac{1}{2}H_2 + \frac{3}{4}H_3 + \frac{3}{4}H$  and  $\{H_i, E_i, F_i\}_{i=1,2}$  define the embedding of  $\mathbf{u}(3)$  in  $\mathbf{u}(4)$  that we want to study; from relations (3) we have that  $\delta_{\mathbb{R}}(\mathbf{u}(3)) \subset \mathbf{u}(3) \wedge \mathbf{u}(3)$  so that  $U(3)$  is a Poisson Lie subgroup.

Let us fix on  $\mathbb{S}^7 = U(3) \backslash U(4)$  the coinduced Poisson bracket  $(\mathbb{S}^7, \{, \})$ . The bracket on  $\mathbb{S}^7$  can be written as the restriction of the following bracket in  $\mathbb{C}^4$ : if  $z_i$ ,  $i = 1, \dots, 4$ , denote complex coordinates we let

$$\begin{aligned} \{z_i, z_j\} &= z_i z_j, \quad 1 \leq i < j \leq 4 & \{z_i^*, z_j^*\} &= -z_i^* z_j^*, \quad 1 \leq i < j \leq 4 \\ \{z_i, z_j^*\} &= -z_i z_j^*, \quad 1 \leq i \neq j \leq 4 & \{z_j^*, z_j\} &= \sum_{i < j} z_j z_j^*. \end{aligned}$$

More detailed information can be found in [30].

The Lie algebra of the diagonal  $SU(2)^d$  is  $\mathbf{su}(2)^d = \langle H_1 + H_3, E_1 + E_3, F_1 + F_3 \rangle$ ; using (3) it is easy to verify that  $\delta_{\mathbb{R}}(\mathbf{su}(2)^d) \not\subset \mathbf{su}(2)^d \wedge \mathbf{u}(4)$  so that  $SU(2)^d$  is not a coisotropic subgroup. We then have to solve the following problem:

*Does there exist any  $g \in U(4)$  such that  $\delta_{\mathbb{R}}(\text{Ad}_g(\mathbf{su}(2)^d)) \subset \text{Ad}_g(\mathbf{su}(2)^d) \wedge \mathbf{u}(4)$ , i.e. such that  $gSU(2)^d g^{-1}$  is coisotropic ?*

We give a positive answer to this question. By direct computation we verify that if

$$g = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix} \in U(4) ,$$

then  $SU(2)_g^d = gSU(2)^d g^{-1}$  is a coisotropic subgroup of  $U(4)$ . This is not the only solution but the general problem will be studied elsewhere. The projection onto this subgroup is then defined by

$$r_g(t) = \begin{pmatrix} \alpha & \beta & 0 & 0 \\ -\beta^* & \alpha^* & 0 & 0 \\ 0 & 0 & \alpha^* & \beta^* \\ 0 & 0 & -\beta & \alpha \end{pmatrix} , \quad |\alpha|^2 + |\beta|^2 = 1 . \quad (4)$$

The right action of  $SU(2)_g^d$  on  $\mathbb{S}^7 \simeq U(3) \backslash U(4)$  is free and defines a principal bundle on  $\mathbb{S}^4 \simeq U(3) \backslash U(4) / SU(2)_g^d$  which is isomorphic to the Hopf bundle. Indeed it is easy to verify that  $i : \mathbb{S}^7 \rightarrow \mathbb{S}^7$ ,  $i(z_1, z_2, z_3, z_4) = (z_1, z_2, -z_4, z_3)$  is a bundle morphism. Nevertheless since  $SU(2)_g^d$  is coisotropic on  $U(4)$ , thanks to Proposition 2, a Poisson structure is coinduced on the base and the projection  $\mathbb{S}^7 \rightarrow \mathbb{S}^4$  is a Poisson map. We call this bundle a *Poisson principal bundle*.

The Poisson structure can be explicitly described by the restriction of the bracket of  $\mathbb{S}^7$  to the subalgebra generated by the following coinvariant functions

$$R = z_1 z_1^* + z_2 z_2^*, \quad a = z_1 z_4^* - z_2 z_3^*, \quad b = z_1 z_3 + z_2 z_4,$$

which satisfy  $|a|^2 + |b|^2 = R(1 - R)$ . Easy calculations prove that:

$$\begin{aligned} \{a, R\} &= -2aR, & \{b, R\} &= 2bR, & \{a, b\} &= -3ab, & \{a, b^*\} &= ab^*, \\ \{a, a^*\} &= -2aa^* + 2R^2, & \{b, b^*\} &= 4bb^* - 2R. \end{aligned}$$

This Poisson algebra has clearly zero rank in  $R = 0$ . Let  $R \neq 0$  and define  $\zeta_1 = a/R$ ,  $\zeta_2 = b/R$ . Geometrically we're just giving cartesian coordinates on the stereographic projection on  $\mathbb{C}^2$ . Poisson brackets between these new coordinates are given by:

$$\begin{aligned} \{\zeta_1, \zeta_2\} &= \zeta_1 \zeta_2, & \{\zeta_1, \zeta_1^*\} &= 2(1 + |\zeta_1|^2), \\ \{\zeta_1, \zeta_2^*\} &= \zeta_1 \zeta_2^*, & \{\zeta_2, \zeta_2^*\} &= -2(1 + |\zeta_1|^2 + |\zeta_2|^2). \end{aligned} \tag{5}$$

Such brackets define a symplectic structure on the 4-dimensional real space  $\mathbb{R}^4$  (it can be proven, in fact that the corresponding map between cotangent and tangent bundle has fixed maximal rank). The covariant Poisson bracket on  $\mathbb{S}^4$  has thus a very simple foliation given by a 0-dimensional leaf and a 4-dimensional linear space.

We summarize this discussion in the following Proposition.

**Proposition 3** *The embedding of  $SU(2)_g^d$  into  $U(4)$  defines a coisotropic subgroup. The corresponding bundle  $\mathbb{S}^7 \rightarrow \mathbb{S}^4 \simeq \mathbb{S}^7 / SU(2)_g^d$  is a Poisson bundle.*

## 5 The Quantum $\Sigma_q^4$

The Hopf algebra  $U_q(4)$  is generated by  $\{t_{ij}\}_{ij=1}^4$ ,  $D_q^{-1}$  and the following relations (see [16]):

$$\begin{aligned} t_{ik} t_{jk} &= q t_{jk} t_{ik}, & t_{ki} t_{kj} &= q t_{kj} t_{ki}, & i < j \\ t_{i\ell} t_{jk} &= t_{jk} t_{i\ell}, & i < j, k < \ell \\ t_{ik} t_{j\ell} - t_{j\ell} t_{ik} &= (q - q^{-1}) t_{jk} t_{i\ell}, & i < j, k < \ell \\ D_q D_q^{-1} &= D_q^{-1} D_q = 1, \end{aligned}$$

where  $D_q = \sum_{\sigma \in P_4} (-q)^{\ell(\sigma)} t_{\sigma(1)1} \dots t_{\sigma(4)4}$  with  $P_4$  being the group of 4-permutations, is central. The Hopf algebra structure is

$$\begin{aligned} \Delta(t_{ij}) &= \sum_k t_{ik} \otimes t_{kj} , & \Delta(D_q) &= D_q \otimes D_q , \\ \epsilon(t_{ij}) &= \delta_{ij} , & \epsilon(D_q) &= 1 , \\ S(t_{ij}) &= (-q)^{i-j} \sum_{\sigma \in P_3(j)} (-q)^{\ell(\sigma)} t_{\sigma(1)1} \dots t_{\sigma(i)i+1} \dots t_{\sigma(4)4} D_q^{-1} , \end{aligned}$$

the compact real structure forces us to choose  $q \in \mathbb{R}$  and is defined by  $t_{ij}^* = S(t_{ji})$ ,  $D_q^* = D_q^{-1}$ .

Let  $L = \text{Span}\{t_{j4}, t_{4j}, t_{44} - 1\}_{j=1,2,3}$ ; it comes out that  $\mathcal{L} = U_q(4)L$  is a Hopf ideal, so that  $U_q(4)/\mathcal{L}$  is equivalent to  $U_q(3)$  as a Hopf algebra. Let  $\ell : U_q(4) \rightarrow U_q(4)/\mathcal{L} \simeq U_q(3)$  be the quotient projection. The algebra  $\mathbb{S}_q^7 = {}^\ell U_q(4)$  is generated by  $z_i = t_{4i}$ ,  $i = 1 \dots 4$ , with the following relations [30]:

$$\begin{aligned} z_i z_j &= q z_j z_i \quad (i < j) , & z_j^* z_i &= q z_i z_j^* \quad (i \neq j) , \\ z_k^* z_k &= z_k z_k^* + (1 - q^2) \sum_{j < k} z_j z_j^* , & \sum_{k=1}^4 z_k z_k^* &= 1 . \end{aligned}$$

The  $U_q(4)$ -coaction on  $\mathbb{S}_q^7$  reads  $\Delta(z_i) = \sum_j z_j \otimes t_{ji}$ .

Let us now quantize the coisotropic subgroup  $SU(2)_g^d$  of Proposition 3. Motivated by the projection  $r_g$  in (4) let us define  $\mathcal{R} = R U_q(4)$ , where

$$\begin{aligned} R &= \text{Span}\{t_{13}, t_{31}, t_{14}, t_{41}, t_{24}, t_{42}, t_{23}, t_{32}, t_{11} - t_{44}, t_{12} + t_{43}, \\ &\quad t_{21} + t_{34}, t_{22} - t_{33}, t_{11}t_{22} - q t_{12}t_{21} - 1\} \\ &= \dot{R} \oplus \text{Span}\{t_{11}t_{22} - q t_{12}t_{21} - 1\} . \end{aligned}$$

It is easy to verify that  $\mathcal{R}$  is a  $\tau$ -invariant, right ideal, two sided coideal. Let  $r : U_q(4) \rightarrow U_q(4)/\mathcal{R}$  be the projection map. We have the following result:

**Proposition 4** *As a  $\tau$ -coalgebra  $U_q(4)/\mathcal{R}$  is isomorphic to  $SU_q(2)$ .*

*Proof.* We sketch here the main lines of the proof. Let  $A_q(N)$  be the bialgebra generated by the  $\{t_{ij}\}$ . We first remark that  $r(D_q) = 1$  so that  $U_q(4)/\mathcal{R} \simeq A_q(4)/\dot{R}A_q(4)$ . First one can show that  $A_q(4)/\dot{R}A_q(4) \simeq A_q(2)$ . Once chosen an order in the generators  $t_{ij}$  of  $A_q(4)$ , a linear basis is given by the ordered monomials in  $t_{ij}$  [17], so that  $A_q(4) = \text{Span}\{t_{11}^{n_{11}} t_{44}^{n_{44}} t_{12}^{n_{12}} t_{43}^{n_{43}} t_{21}^{n_{21}} t_{34}^{n_{34}} t_{22}^{n_{22}} t_{33}^{n_{33}}\}$ . Making a repeated use of the following relations for  $i < k, j < l$

$$\begin{aligned} t_{ij}^n t_{kl} &= t_{kl} t_{ij}^n - q^{-1}(1 - q^{2n}) t_{il} t_{kj} t_{ij}^{n-1} , \\ t_{ij}^m t_{kl}^n &= t_{kl}^m t_{ij}^n + q(1 - q^{-2m}) t_{il} t_{kj} t_{kl}^{m-1} , \end{aligned}$$

we get that  $A_q(4)/\dot{R}A_q(4) = \text{Span}\{t_{11}^{n_{11}} t_{12}^{n_{12}} t_{21}^{n_{21}} t_{22}^{n_{22}}\} \simeq A_q(2)$ . To show that this a  $\tau$ -coalgebra isomorphism is equivalent to verify that the projection  $r$  restricted to the first quadrant of  $A_q(4)$  is a  $\tau$ -bialgebra isomorphism. This can be directly done by using the relations. Finally the quotient by the quantum determinant gives  $SU_q(2)$ . ■

**Remark 5** The projection map  $r : U_q(4) \rightarrow SU_q(2)$  is not a Hopf algebra map as can be, for instance, explicitly verified on  $r(t_{11}t_{43}) \neq r(t_{11})r(t_{43})$ .

In the following we will denote  $U_q(4)/\mathcal{R}$  with  $SU_q(2)$ , but we have to be careful that  $r$  doesn't preserve the algebra structure but only defines a right  $U_q(4)$ -module structure on the quotient.

By construction  $\Delta_r = (\text{id} \otimes r)\Delta : \mathbb{S}_q^7 \rightarrow \mathbb{S}_q^7 \otimes SU_q(2)$  defines an  $SU_q(2)$  coaction on  $\mathbb{S}_q^7$ . The space of functions on the quantum 4-sphere  $\Sigma_q^4$  is the space of coinvariants with respect to this coaction, *i.e.*  $\Sigma_q^4 = \{a \in \mathbb{S}_q^7 \mid \Delta_r(a) = a \otimes r(1)\}$ . We describe  $\Sigma_q^4$  in the following proposition whose proof is postponed in the Appendix.

**Proposition 6** *The algebra  $\Sigma_q^4$  is generated by  $\{a, a^*, b, b^*, R\}$ , where  $a = z_1 z_4^* - z_2 z_3^*$ ,  $b = z_1 z_3 + q^{-1} z_2 z_4$ ,  $R = z_1 z_1^* + z_2 z_2^*$ . They satisfy the following relations*

$$\begin{aligned} Ra &= q^{-2}aR, & Rb &= q^2bR, & ab &= q^3ba, & ab^* &= q^{-1}b^*a, \\ aa^* + q^2bb^* &= R(1 - q^2R), \\ aa^* &= q^2a^*a + (1 - q^2)R^2, & b^*b &= q^4bb^* + (1 - q^2)R. \end{aligned}$$

In terms of  $r_{ij} = r(t_{ij}) \in SU_q(2)$ , with  $i, j = 1, 2$ , the fundamental corepresentation  $\tau_f : \mathbb{C}^2 \rightarrow SU_q(2) \otimes \mathbb{C}^2$  is written as

$$\tau_f \begin{pmatrix} e_1 \\ e_2 \end{pmatrix} = \begin{pmatrix} r_{11} & r_{12} \\ r_{21} & r_{22} \end{pmatrix} \otimes \begin{pmatrix} e_1 \\ e_2 \end{pmatrix}.$$

Let  $\mathcal{E} = \mathbb{S}_q^7 \square_{\tau_f} \mathbb{C}^2$  the associated quantum vector bundle. Let  $\langle (a_1, a_2), (b_1, b_2) \rangle = a_1 b_1^* + a_2 b_2^* \in \Sigma_q^4$  for  $(a_1, a_2), (b_1, b_2) \in \mathcal{E}$  be the hermitian structure in  $\mathcal{E}$ . Let

$$f_1 = q(z_1, z_2), \quad f_2 = q(z_2^*, -qz_1^*), \quad f_3 = (z_4, -z_3), \quad f_4 = q(z_3^*, q^{-1}z_4^*),$$

and  $G \in M_4(\Sigma_q^4)$  such that  $G_{ij} = \langle f_i, f_j \rangle$ . We then have the following description of  $\mathcal{E}$  (see the Appendix for the proof).

**Proposition 7** *As a  $\Sigma_q^4$ -module  $\mathcal{E}$  is generated by  $f_i$ ,  $i = 1 \dots 4$ ; it is isomorphic to  $(\Sigma_q^4)^4 G$  where*

$$G = G^2 = \begin{pmatrix} q^2R & 0 & qa & q^2b \\ 0 & q^2R & qb^* & -q^3a^* \\ qa^* & qb & 1 - R & 0 \\ q^2b^* & -q^3a & 0 & 1 - q^4R \end{pmatrix}. \quad (6)$$

## 6 Unitary Representations of $\Sigma_q^4$

Let  $0 < q < 1$ . By restriction of those of  $\mathbb{S}_q^7$ , see for instance [7], we obtain the following two inequivalent unitary representations of  $\Sigma_q^4$ . The first is one dimensional and it is obtained as the restriction of the counit  $\epsilon$  of  $U_q(4)$ :

$$\epsilon(R) = \epsilon(a) = \epsilon(b) = 0. \quad (7)$$

The second one  $\sigma : \Sigma_q^4 \rightarrow B(\ell^2(\mathbb{N})^{\otimes 2})$  is defined by

$$\begin{aligned} \sigma(R)|n_1, n_2\rangle &= q^{2(n_1+n_2)}|n_1, n_2\rangle, \\ \sigma(a)|n_1, n_2\rangle &= q^{n_1+2n_2-1}(1-q^{2n_1})^{1/2}|n_1-1, n_2\rangle, \\ \sigma(b)|n_1, n_2\rangle &= q^{n_1+n_2}(1-q^{2(n_2+1)})^{1/2}|n_1, n_2+1\rangle. \end{aligned} \quad (8)$$

There are no other irreducible representations with bounded operators. In fact let  $\rho : \Sigma_q^4 \rightarrow B(\mathcal{H})$  be such a representation, since  $\rho(R)$  is a bounded selfadjoint operator and  $Ra = q^{-2}aR$ ,  $Rb^* = q^{-2}b^*R$ , there exists a vector  $|\lambda\rangle$  such that  $\rho(R)|\lambda\rangle = \lambda|\lambda\rangle$  and  $\rho(a)|\lambda\rangle = \rho(b^*)|\lambda\rangle = 0$ . By using the relation  $a^*a + bb^* = q^{-2}R(1-R)$  we conclude that  $\lambda = 0$  or  $\lambda = 1$ . Being  $\rho$  irreducible it can be verified that for  $\lambda = 0$  we have that  $\rho = \epsilon$  and for  $\lambda = 1$  we have  $\rho = \sigma$ .

Let us remark that such irreducible (unitary) representations are in one to one correspondence with the leaves of the symplectic foliation of the underlying Poisson 4-sphere: the 0-dimensional leaf corresponds to the counit and the symplectic  $\mathbb{R}^4$  to the infinite dimensional representation. The representation  $\sigma$  has the following important property.

**Proposition 8** *The operator  $\sigma(x) \in B(\ell^2(\mathbb{N})^{\otimes 2})$  is a trace class operator for each  $x \in \bar{\Sigma}_q^4 = \Sigma_q^4/\mathbb{C}1$ .*

*Proof.* Since the family of trace class operators  $\mathcal{I}_1$  is a  $*$ -ideal in the algebra of bounded operators, it is enough to verify the proposition on the generators  $\sigma(R)$ ,  $\sigma(a)$  and  $\sigma(b)$ . Indeed we have that  $\text{tr}(\sigma(|R|)) = \text{tr}(\sigma(R)) = (1-q^2)^{-2}$  and  $\text{tr}(\sigma(|a|)) = \sum_{n_1, n_2 \geq 0} q^{n_1+2n_2-1}(1-q^{2n_1})^{1/2} = q^{-1}(1-q^2)^{-1} \sum_{n \geq 0} q^n(1-q^{2n})^{1/2} \leq q^{-1}(1-q^2)^{-1} \sum_{n \geq 0} q^n(1-q^{2n}) = (1-q)^{-1}(1-q^3)^{-1}$ . Analogously  $\text{tr}(|b|) \leq (1+q^2)(1-q)^{-1}(1-q^3)^{-1}$ . ■

**Remark 9** The universal  $C^*$ -algebra  $C(\Sigma_q^4)$ , defined by  $\Sigma_q^4$ , is the norm closure of  $\sigma(\Sigma_q^4)$ . By Proposition 8 we have that  $\sigma(\Sigma_q^4) \setminus \mathbb{C}1$  is contained in the algebra  $K$  of compact operators on  $B(\ell^2(\mathbb{N})^{\otimes 2})$ . Using Proposition 15.16 of [13] we conclude that  $C(\Sigma_q^4)$  is isomorphic to the unitization of compacts.

Note that, though different at an algebraic level, it is not possible to distinguish from their  $C^*$ -algebras our 4-sphere and the standard Podleś sphere  $\mathbb{S}_q^2(c, 0)$  in [22, 27]. A possible explanation for this peculiarity stands in the fact that the space of leaves of the underlying symplectic foliations are homeomorphic.

Let  $H = \ell^2(\mathbb{N})^{\otimes 2} \oplus \ell^2(\mathbb{N})^{\otimes 2}$  and  $\pi = \begin{pmatrix} \sigma & 0 \\ 0 & \epsilon \end{pmatrix}$ ,  $\gamma = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ . As a consequence of Proposition 8 we have that  $(H, \pi)$  is a 0-summable Fredholm module whose character is  $\text{tr}_\sigma = \text{tr}(\sigma - \epsilon)$ . By explicit computation we have  $\text{tr}_\sigma(1) = 0$  and

$$\begin{aligned} \text{tr}_\sigma(R^k) &= \frac{1}{(1 - q^{2k})^2} \quad (k > 0), \quad \text{tr}_\sigma(a) = \text{tr}_\sigma(b) = 0, \\ \text{tr}_\sigma(aa^*) &= \frac{1}{(1 - q^4)^2}, \quad \text{tr}_\sigma(bb^*) = \frac{1}{(1 - q^2)(1 - q^4)}. \end{aligned} \quad (9)$$

In the representation  $\sigma$ ,  $R$  is an invertible operator. This suggests the possibility of realizing a *quantum stereographic transformation* on a deformation of  $\mathbb{C}^2$ , that we will denote  $\mathbb{C}_q^2$ . Let us define  $\zeta_1 = R^{-1}a$ ,  $\zeta_2 = bR^{-1}$ ; by direct computation we find that they satisfy the following relations:

$$\begin{aligned} \zeta_1\zeta_2 &= q^{-1}\zeta_2\zeta_1, & \zeta_1\zeta_1^* &= q^{-2}\zeta_1^*\zeta_1 + (1 - q^2), \\ \zeta_1\zeta_2^* &= q^{-1}\zeta_2^*\zeta_1, & \zeta_2\zeta_2^* &= q^2\zeta_2^*\zeta_2 - (1 - q^2)(q^2 + \zeta_1^*\zeta_1). \end{aligned} \quad (10)$$

One can verify that the algebra  $\mathbb{C}_q^2$  quantizes the symplectic structure on  $\mathbb{C}^2$  seen in (5).

## 7 Chern–Connes paring of $\mathbf{G}$

Let us compute the Chern classes in cyclic homology associated to the projector  $G$ . We briefly recall some basic definitions and results from cyclic homology, see [9] and [19] for any details.

Let  $A$  be an associative  $\mathbb{C}$ -algebra. Let  $d_i(a_0 \otimes a_1 \dots \otimes a_n) = a_0 \otimes \dots a_i a_{i+1} \dots \otimes a_n$ , for  $i = 0, \dots, n-1$  and  $d_n(a_0 \otimes a_1 \dots \otimes a_n) = a_n a_0 \otimes a_1 \dots \otimes a_{n-1}$ ; the Hochschild boundary is defined as  $\beta = \sum_{i=0}^n (-)^i d_i$  and the Hochschild complex is  $(C_*(A), \beta)$ , with  $C_n(A) = A^{\otimes n+1}$ . As usual we denote Hochschild homology with  $HH_*(A)$ .

Let  $t(a_0 \otimes a_1 \otimes \dots \otimes a_n) = (-)^n a_1 \otimes \dots \otimes a_n \otimes a_0$  be the cyclic operator and  $C_n^\lambda(A) = A^{\otimes n+1}/(1-t)A^{\otimes n+1}$ . The Connes complex is then  $(C_*^\lambda(A), \beta)$ ; its homology is denoted as  $H_*^\lambda(A)$ . For each projector  $G \in M_k(A)$ , i.e.  $G^2 = G$ , the Chern class is defined as  $ch_n^\lambda(G) = \text{Tr}[(-)^n G^{\otimes 2n+1}] \in H_{2n}^\lambda(A)$ , where  $\text{Tr} : M_k(A)^{\otimes n} \rightarrow A^{\otimes n}$  is the generalized trace, i.e.  $\text{Tr}[M_1 \otimes \dots \otimes M_n] = \sum_j [M_1]_{j_1 j_2} \otimes [M_2]_{j_2 j_3} \dots \otimes [M_n]_{j_n j_1}$ .

Let  $I : A^{\otimes n+1} \rightarrow C_n^\lambda(A)$  be the projection map and let  $x \in A^{\otimes n}$  be such that  $I(x)$  is a cycle which induces  $[I(x)] \in H_n^\lambda$ . Let us define the periodicity map  $S : H_n^\lambda \rightarrow H_{n-2}^\lambda$ , as

$$S([I(x)]) = -\frac{1}{n(n-1)} \left[ I \left( \sum_{0 \leq i < j \leq n} (-)^{i+j} d_i d_j(x) \right) \right].$$

There is then a long exact sequence in homology:

$$\dots \rightarrow HH_n(A) \xrightarrow{I} H_n^\lambda(A) \xrightarrow{S} H_{n-2}^\lambda \xrightarrow{B} HH_{n-1}(A) \dots, \quad (11)$$

where  $B$  is an operator we don't need to define. With our normalization of the Chern character, we have that for each projector  $G$

$$S(ch_n^\lambda(G)) = -\frac{1}{2(2n-1)} ch_{n-1}^\lambda(G) . \quad (12)$$

Let  $G \in M_4(\Sigma_q^4)$  be the projector defined in Proposition 7. Then

$$ch_0^\lambda(G) = [\text{Tr}(G)] = [2 - (1 - q^2)^2 R] \in H_0^\lambda .$$

The character  $\text{tr}_\sigma$  of the Fredholm module  $(H, \pi)$  given in (9) is a well defined cyclic 0-cocycle on  $\Sigma_q^4$ . We then have that  $\text{tr}_\sigma(ch_0^\lambda(G)) = -1$  and conclude that  $ch_0^\lambda(G)$  defines a non trivial cyclic cycle in  $H_0^\lambda(\Sigma_q^4)$ ; using the  $S$ -operator (12) and Connes sequence (11) we conclude that  $ch_1^\lambda$  and  $ch_2^\lambda$  define non trivial classes in cyclic homology and are not Hochschild cycles. Since  $\text{tr}_\sigma$  is the character of a Fredholm module, the integrality of the pairing is a manifestation of the so called noncommutative index theorem [9]. We summarize this discussion in the following proposition.

**Proposition 10** *The projector  $G$  defined in (6) defines non trivial cyclic homology classes  $ch_n^\lambda(G) \in H_{2n}^\lambda$ . The Chern-Connes pairing with  $\text{tr}_\sigma$  defined in (9) is:*

$$\langle \text{tr}_\sigma, G \rangle = -1 .$$

## Appendix. Proof of Proposition 6 and 7

To prove Proposition 6 and 7 we use the strategy adopted by Nagy in [24]. His argument is based on the general fact that the corepresentation theory of compact quantum groups is “equivalent” to the classical one. This equivalence is realized by a bijective map between quantum and classical finite dimensional corepresentations: this map preserves direct sum, tensor product and dimension. The fact that we deal with coisotropic subgroups requires some additional care. The projection  $r$  induces a mapping  $r[\rho] = (\text{id} \otimes r)\rho$  from the corepresentations of  $U_q(4)$  into the corepresentations of  $SU_q(2)$ , since  $r$  is not an algebra morphism it is not obvious that this mapping preserve the tensor product. However in our case the following lemma can be proved:

**Lemma 11** *Let  $t_f$  and  $t_{fc}$  be the fundamental and its contragredient corepresentation of  $U_q(4)$ , then  $r[t_f^{\otimes r} \otimes t_{fc}^{\otimes s}]$  is equivalent to  $r[t_f]^{\otimes r} \otimes r[t_{fc}]^{\otimes s}$ .*

*Proof.* Let  $\tau_f$  be the fundamental corepresentation of  $SU_q(2)$  and  $\tau_{fc}$  its contragredient. Let us notice that for  $i, j = 1, 2$  we have that  $r[t_{i,j}] = (\tau_f)_{ij}$ ,  $r[t_{i+2,j+2}] = q^{i-j}(\tau_{fc})_{ij}$ ,  $r[t_{i,j}^*] = (\tau_{fc})_{ij}$  and  $r[t_{i+2,j+2}^*] = q^{j-i}(\tau_f)_{ij}$ . By making use of the equivalence between  $\tau_f$  and  $\tau_{fc}$  it is easy to conclude that  $r[t_f^{\otimes r} \otimes t_{fc}^{\otimes s}]$  for  $r + s = 2$  is

equivalent to  $4 \tau_f^{\otimes 2}$  and then to  $r[t_f]^{\otimes r} \otimes r[t_{fc}]^{\otimes s}$ . The result for generic  $r$  and  $s$  is obtained by recurrence and by making use of the right  $U_q(4)$ –module structure of the projection  $r$ . ■

As a consequence the decomposition of  $r[t_f^{\otimes n}]$  into irreducible corepresentations of  $SU_q(2)$  is the same as the classical one.

Let  $t_n = \bigoplus_{r+s \leq n} t_{r,s}$  where  $t_{r,s} = t_f^{\otimes r} \otimes t_{fc}^{\otimes s}$ . We denote with  $C(t_n) \subset U_q(4)$  the subcoalgebra of the matrix elements of  $t_n$ . We then have  $U_q(4) = \bigcup_{n \in \mathbb{N}} C(t_n)$  and we define  $\mathbb{S}_{q,n}^7 = C(t_n) \cap \mathbb{S}_q^7$ . Obviously  $\mathbb{S}_{q,n}^7$  is a  $U_q(4)$ –comodule with coaction  $\Delta_n = \Delta|_{\mathbb{S}_{q,n}^7}$ . From the decomposition into irreducible corepresentations  $\Delta_n = \sum_{\lambda \in I(U_q(4))} m_\lambda \lambda$ , where  $m_\lambda \in \mathbb{N}$ , we get  $\mathbb{S}_{q,n}^7 = \bigoplus_{j=1, \dots, m_\lambda} \mathbb{S}_{q,n}^{7\lambda, j}$ .

Let  $\rho : V \rightarrow SU_q(2) \otimes V$  be an irreducible  $SU_q(2)$  corepresentation. We prove the following Lemma.

**Lemma 12** *The dimension of  $\mathbb{S}_{q,n}^7 \square_\rho V$  doesn't depend on  $q$ .*

*Proof.* Let  $P_\rho : \mathbb{S}_q^7 \otimes V \rightarrow \mathbb{S}_q^7 \square_\rho V$  be the projection defined by  $P_\rho(f \otimes v) = \sum_{(f,v)} f_{(0)} h(f_{(1)} S(v_{(-1)})) \otimes v_{(0)}$ , where  $h$  is the Haar measure on  $SU_q(2)$ . We obviously have that  $P_\rho(\mathbb{S}_{q,n}^7 \otimes V) = \bigoplus_{j=1, \dots, m_\lambda} P_\rho(\mathbb{S}_{q,n}^{7\lambda, j} \otimes V)$ . Then  $\dim P_\rho(\mathbb{S}_{q,n}^7 \otimes V) = \sum_{\lambda \in I(U_q(4))} m_\lambda m_\rho(\lambda)$ , where  $m_\rho(\lambda) = \dim P_\rho(\mathbb{S}_{q,n}^{7\lambda, j} \otimes V)$  equals the multiplicity of  $\rho$  in the decomposition of  $r[\lambda] = (\text{id} \otimes r) \lambda$ . Since the correspondence between classical and quantum corepresentation preserves dimensions, the result follows. ■

*Proof of Proposition 6.* To show that  $\{a, b, R\}$  are coinvariants is a direct computation. Let  $B_q \subset \Sigma_q^4$  be the  $*$ –algebra generated by those elements. By the use of the diamond lemma the monomials  $\{a^{*i_1} a^{i_2} R^j b^{*k_1} b^{k_2} \mid k_1 k_2 = 0\}$  are linearly independent and they form a basis of  $B_q$ . Note that the same monomials form a basis for the polynomial functions on the classical 4–sphere, and define a vector space isomorphism which maps  $B_{q,n} = C(t_n) \cap B_q \rightarrow P_{\mathcal{K}}(\mathbb{S}_{1,n}^7)$ , where  $P_{\mathcal{K}} = P_\rho$  with  $\rho$  being the identity corepresentation. Using the Lemma 12 we then have  $B_q = \Sigma_q^4$ . ■

*Proof of Proposition 7.* By a direct check it is easy to see that  $f_i$  are in  $\mathcal{E} = \mathbb{S}_q^7 \square_{\tau_f} \mathbb{C}^2$  and that the mapping  $f_i \rightarrow e_i G$ , with  $(e_i)_j = \delta_{ij}$ , is a  $\Sigma_q^4$ –module morphism. Since in the classical case it is clearly bijective the result follows by repeating the same arguments of the proof of Proposition 6 and applying the Lemma 12 with  $\rho = \tau_f$ . ■

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